Building Blocks of Symmetry

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Symmetry is a concept used in many different contexts from art to science. In mathematics symmetry is rigorously defined and the abstract notion has many different concrete mathematical instances. Recent classification of the building blocks of finite symmetries is a monumental achievement of joint work of several mathematicians. Here we define the notion of symmetry, briefly introduce simple groups and review some results from the classification. Some sporadic groups, symmetries of mind-blowing combinatorial objects, will also be discussed.

1 Symmetry

The everyday notion of symmetry has a very vague and somewhat limited meaning: balanced, well-proportioned, harmony between the parts. More specifically, we often mean only bilateral symmetry. We say that things and animals are symmetrical when they have parts that are mirror images of each other. The prime example is the human body.

The mathematical notion incorporates these symmetries but the definition is lot more general: symmetry is defined via operations, transformations that leave some aspects of the transformed object unchanged. Here are some variations of this definition from leading researchers of mathematical symmetry:

> *" ...invariance of a configuration of elements under a group of automorphic transformations."*, Hermann Weyl: Symmetry 1952. [13]

> *"Symmetry is not a number or a shape, but a special kind of transformation – a way to move an object. If the object looks the same after being transformed, then the transformation concerned is a symmetry."*, Ian Stewart: Why Beauty is Truth, 2007. [12]

> *"You could think of the total symmetry of an object as all the moves that the mathematician could make to trick you into thinking that he hadn't touched it at all."*, Marc Du Sautoy: Finding Moonshine: A Mathematician's Journey Through Symmetry 2008. [5]

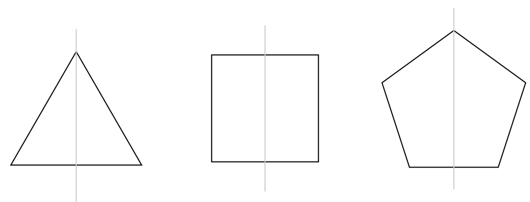

Figure 1: Regular polygons have symmetry groups from the same family, the *dihedral groups*. These groups contain a mirror symmetry on the axes indicated and the $\frac{2\pi}{n}$ clockwise rotations.

Therefore something is symmetrical if there is a special kind of operation defined on it. Thus *symmetry* becomes some sort of transformation, movement instead of some static property. These symmetry transformations can be composed by simply executing them one after the other yielding other symmetry operations. We call a set of these transformations that is closed under the composition a *group*.

> *"Numbers measure size, groups measure symmetry."*, M.A. Armstrong: Groups and Symmetry 1988. [1]

By measuring we ususally mean assigning a number to an object. For example,

$$\begin{aligned} \text{unit sphere } (r=1) &\longmapsto 4\pi, \\ \text{integer number} &\longmapsto \text{number of divisors}, \\ \text{continuous function} &\longmapsto \text{number of local minima}. \end{aligned}$$

Looking at these mappings more abstractly,

$$\text{object} \longmapsto \text{value}$$

we can conclude that measurement values can be of different types, like integers and real numbers in the above examples. But why stopping here? We can assign to objects more complex measurement values, like structured sets. As far as they capture some key properties of the objects, we can call all these maps measurements. For promoting the idea of measurements with general algebraic objects see [11].

For instance we can measure how symmetric the regular polygons are by their *symmetry groups* (Fig. 1). Similarly for regular polyhedra (Fig. 2), and of course for higher dimensional regular objects.

We can also measure the symmetry of some combinatorial objects if the symmetry operation is some rearrangement of the elements. A function $p : X \to X$ on the set X is called a *permutation* if it is one-to-one and onto (therefore it is invertible, a bijection). Examples in cyclic notation: $p = (1,2,3,4,5)$, $t = (1,2)(3,4)$, meaning that $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 4$, $4 \mapsto 5$, $5 \mapsto 1$ under p, and

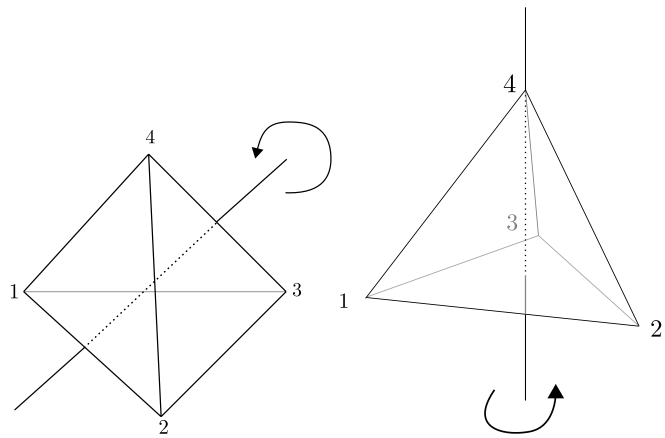

Figure 2: Symmetry operations flip and rotate generate the symmetry group of the tetrahedron, A_4.

$1 \mapsto 2$, $2 \mapsto 1$, $3 \mapsto 4$, $4 \mapsto 3$, $5 \mapsto 5$ under t. Thus a permutation of set X is a symmetry of X. Permutations can naturally be combined just by executing them one after the other $(1,2,3,4,5) \cdot (1,2)(3,4) = (2,4,5)$. The group contains the *identity* 1 and *inverse* map p^{-1} for each element p, thus everything can undone within a group. A *permutation group* is a set G of permutations closed under composition (multiplication, usually denoted by $\cdot$).

The algebraic expression $x_1 + x_2 + x_3 + x_4$ is clearly is invariant under all possible permutations of the set $\{x_1, x_2, x_3, x_4\}$, while $x_1 + x_2 + x_3 \cdot x_4$ only admits swapping x_1, x_2 and x_3, x_4.

1.1 Historical Sources of Group Theory

The emergence of group theory follows a usual pattern. In different branches of mathematics groups independently occured in different contexts, but the common pattern was not recognized immediately. Following the description of [9], the concept of groups appeared in four different fields:

Classical Algebra (Lagrange, 1770) Up to the end of the 18th century algebra was about solving polynomial equations. Lagrange analyzed the existing solutions of cubic and quartic equations and also the general case. He constructed a so called resolvent equation:

1. giving a rational function of the n roots and coefficients of the original equation
2. collecting the distinct values of this rational function when the n roots are permuted, $y_1, \ldots, y_k$
3. the resolvent equation is $(x - y_1)(x - y_2) \ldots (x - y_k)$

He showed that k divides $n!$, which we now know more generally as the Lagrange Theorem, stating that the order of a subgroup divides the order of the group. Lagrange did not mention the group concept explicitly, that appeared only later in Galois' work. The key point is that the symmetries of a mathematical object (here this object is an equation) are studied.

Number Theory (Gauss, 1801) In *Disquisitiones Arithmeticae* groups appear in four different ways: the additive group of integers modulo m, the multiplicative group of integers relatively prime to m, the equivalence classes of binary quadratic forms, and the group of n-th roots of unity. These are all abelian groups, i.e. the group operation is commutative. However, there is no unifying concept, these groups are used only in number-theoretical contexts.

Geometry (Klein, 1874) Among the properties a geometric figure has, we are interested in those that are invariant under some transformation. This way the transformation becomes the primary object of study. In Klein's Erlangen Program he suggested that group theory is a useful way of organizing geometrical knowledge, so he used the group concept explicitly.

Analysis (Lie, 1874; Poincaré and Klein, 1876) Sophus Lie aimed to do similar things to differential equation as Lagrange and Galois did to polynomial equation. Key problem is to find continuous transformation groups that leave analytic functions invariant.

Beyond doubt doing mathematics requires the skill of making abstractions. If we were allowed to characterize mathematics with only one trait, the term abstract would be the right candidate. However, it seems that certain amount of time is needed to make important abstractions. First half of the 19th century mathematics already produced different concrete group examples, but the concept of the abstract group only appeared at the end of 19th century. The early attempt in 1854 by Arthur Cayley, when he actually defined the abstract group as a set with a binary operation, had no recognition by fellow mathematicians.

After the proper abstraction is made new specializations of the theory appear (e.g. finite, combinatorial, infinite abelian, topological group theory).

2 Classifications

Another (theoretical) activity that humans often do is *classification*. If we have many objects we try to classify them, i.e. to put them in classes containing objects of the same sort. First we identify those with some superficial difference (e.g. renaming its components) so they are essentially the same (up to an isomorphism), then we collect those that are members of the same family, maybe differing in their sizes but their structure following the same pattern.

2.1 Finite Abelian Groups

A very easy, exercise level classification is the one of the finite abelian groups.Any finite abelian group is isomorphic to a direct product of cyclic groups (counters) of prime power order. The components are uniquely determined (up to an reordering).

2.2 Wallpaper Patterns

Beautiful patterns can be created by repeating geometric motifs according to some symmetry. Color and the artistic shape of the motif can be varied endlessly but the number of symmetry types are limited and can be fully classified. There are 17 wallpaper symmetry patterns on the plane [13, 4]. Alhambra, the Moorish castle in Granada (Andalusia, Spain) exhibits all these patterns. It is an interesting mathematical challenge for the tourists to find all these patterns (a vivid description of this quest can be found in [5]). Why 17? The answer is a long and subtle proof, but one underlying fact is that there are only a few tile shapes that can fit together to cover the plane. In 3 dimensions there are 230 crystallographic groups.

The classification of wallpaper patterns is complete, so whenever we find a seemingly new pattern we can always figure out eventually which of the seventeen cases (see Fig. 3) it belongs to.

2.3 Finite Simple Groups

One of the most important achievements of mathematics is the classification theorem of finite simple groups, the building blocks of symmetry.

2.3.1 Simple Groups

We usually understand things by taking them apart until basic building blocks are found and we can recognize the ways how these bits can be put together. This is how physics proceeds: from macroscopic objects down to their constituent atoms, then from atoms to elementary particles. Mathematics applies the same method. For instance, for integer numbers the prime numbers are the building blocks, and for building composite numbers we use multiplication, which is repeated addition. Since we use groups for measuring like numbers, we would like to do some similar decomposition theory for groups as well (Figure 4). But what is a building block for a group of symmetries? It has to be a subgroup, i.e. a subset that is closed in respect to the multiplication. Also, it could not be the trivial group (consisting of only the identity) and group itself, just like we exclude 1 and n itself from the factors of the prime decomposition. Moreover, it turns out that not any proper, nontrivial subgroup would do for dividing a group. It has to be a *normal subgroup*. This means that taking the normal subgroup and its translates within the group, and considering these as

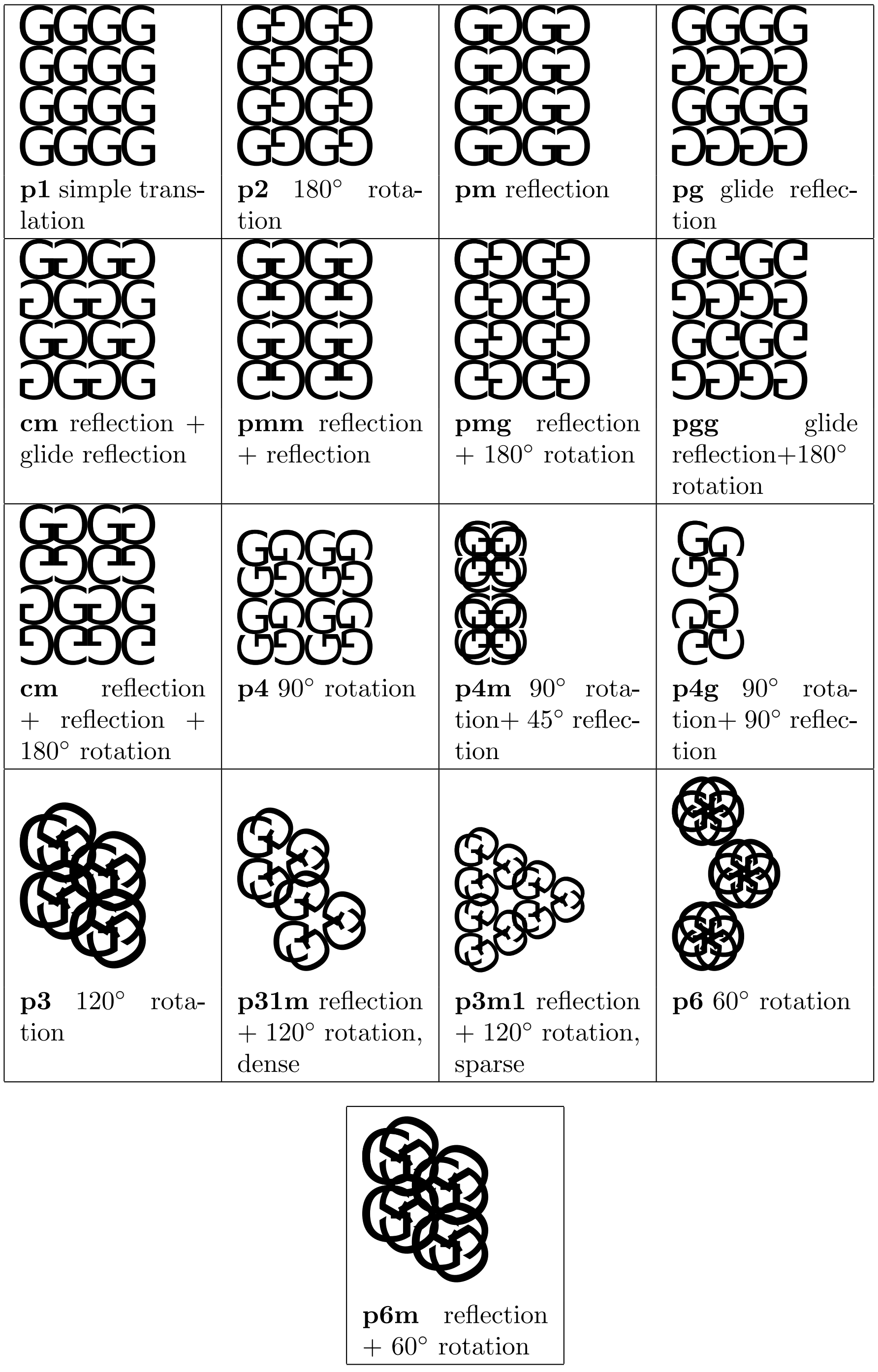

Figure 3: The 17 wall paper symmetry patterns applied to a simple G letter. (The patterns were created by the Inkscape (http://inkscape.org vector graphics editor.)

	Natural Numbers	**Groups**
Building Blocks	Primes	Simple Groups
Composition	Multiplication/Division	Extension/Factoring

Figure 4: The parallel between the prime decomposition of integers and group decompositions.

a new set of points to act on by the original group, we still get a group structure. This is called the *factor group*. Then simple groups are that have no such normal subgroups.

2.3.2 The theorem

Any finite simple group is isomorphic to one of these:

1. A cyclic group with prime order (counters modulo m). These are all abelian.
2. An alternating group of degree at least 5 (permutation groups consisting of all even permutations on 5 or more points).
3. A simple group of Lie type, including both
   (a) the classical Lie groups, namely the groups of projective special linear, unitary, symplectic, or orthogonal transformations over a finite field;
   (b) the exceptional and twisted groups of Lie type (including the Tits group which is not strictly a group of Lie type).
4. One of 26 sporadic simple groups.

In 2004 the last known gap of the proof had been filled, so we now believe that we have the proof for this theorem. However, the proof is not a short one, it is written down in several hundreds of journal papers. There are recent attempts to summarize the proof and to bring the topic down to textbook level [3, 14]. It is probably not an exaggeration to say that even in the 21st century there is some danger in loosing some mathematical knowledge. Finite group theory is less attractive as it is seemingly "done, finished", therefore researchers and PhD students pursue other research directions and the old generations retire. So despite the well-organized and fully electronic storage of mathematical texts we still may lack the persons capable of understanding them.

2.4 Sporadic Groups

The sporadic groups are not in any families, they are unique and exceptional in every possible sense [2, 7]. Due their size we cannot represent them explicitly, for instance by enumerating all elements, but we usually characterize them as

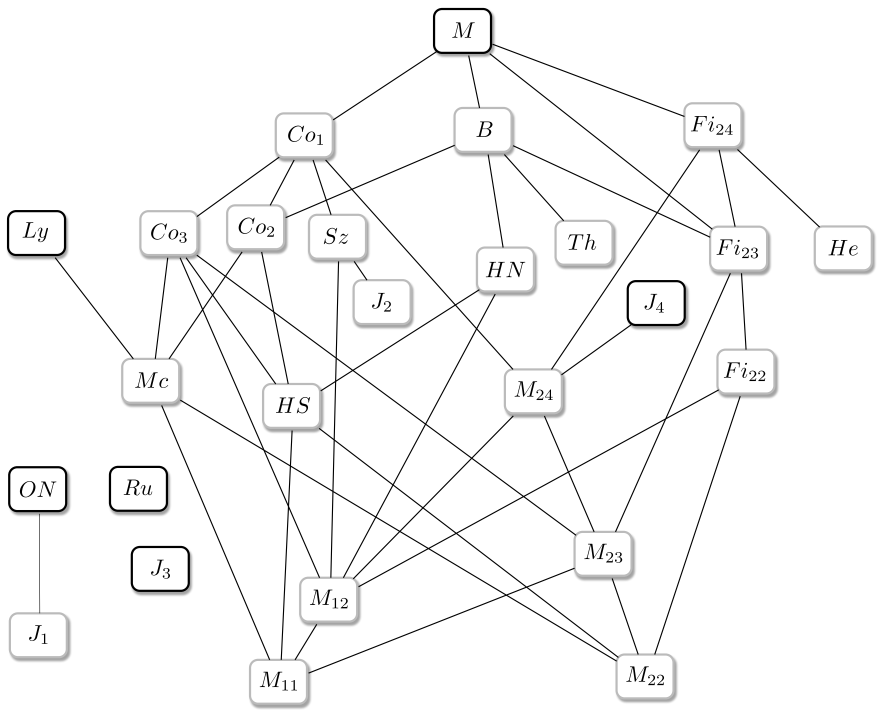

Figure 5: The sporadic groups. The lines indicate the subgroup of relation. Darker shade indicates that a sporadic group is not a subgroup of any other sporadic group.

automorphism groups of some mathematical structures, following the guiding quote of Hermann Weyl:

> *"A guiding principle in modern mathematics is this lesson: Whenever you have to do with a structure-endowed entity S, try to determine its group of automorphisms, the group of those element-wise transformations which leave all structural relations undisturbed. You can expect to gain a deep insight into the constitution of S in this way."*[13]

2.4.1 Witt design – M_{24}

We have 24 symbols and we make 8-tuples, *octads* from them such a way that each set of five symbols, *quintiples*, lies in exactly one octad. Let's count the number of quintiples first. There are $24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 = 5100480$. In each octad the number of quintiples is $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6720$. Let's denote the number of octads by N. Since each quintiples lies in only one octad we have the following equation:

$$N \cdot 6720 = 5100480$$

thus

$$N = \frac{5100480}{6720} = 759$$

If someone is not familiar with design theory, then this might be a little bit surprising since one would expect this number a bit bigger as there are quite many quintiples of 24 symbols. However, one octad contains many quintiples, so it is a very compressed structure. No wonder that this packed combinatorial object has so many symmetries.

2.4.2 Leech Lattice – Sphere Packing in 24 dimensions

Sphere packing is an old problem of mathematics. The aim is to pack more spheres in the given volume. In 2 dimensions it is easy to see the solution Fig. 6 Kepler in 1611 conjectured that the most efficient packing of spheres is exactly how one would arrange oranges in a grocery store, but the final proof by Thomas Hales in 1998 appeared only in 2005 [8]. We are also interested in packing in higher dimensions, not because of higher diemnsional oranges, but because the lattices defined by efficient packing can be used for error correcting codes when transmitting information. Tight packings are hard to find in higher than 8 dimensions, but in dimension 24 something extraordinary happens. Using the Witt design one can construct a lattice in which each 24-dimensional circle touches 196,560 others. In 2 dimensions each circle (2-dimensional spheres) touches 6 neighbours in the tightest packing. The construction is combinatorial, not geometrical anymore. Therefore describing the position of a sphere we need a 24-tuple. Let's consider the neighbours of the sphere placed in the origin (with 24 zero coordinates). The set of touching spheres have 3 subsets:

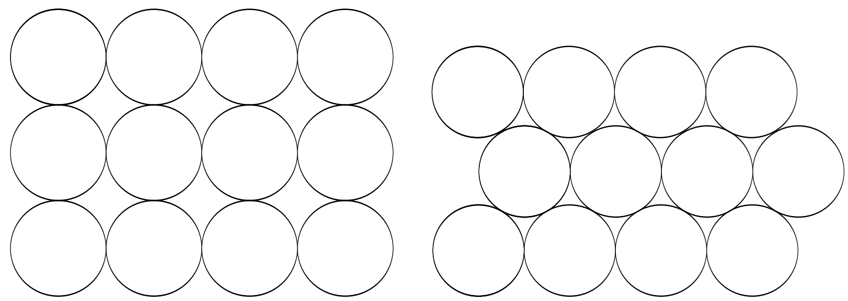

Figure 6: Sphere packing in 2 dimensions. The right pattern is the most efficient packing methond in the plane.

- Taking Witt's design, we put +2 or −2 in the coordinates chosen by the elements of an octad, the parity of negative signs is even, zero elsewhere.

$$2^7 \cdot 759 = 97152$$

- 2 coordinates are +4 or −4 the remaining 22 coordinates are all zero.

$$2^2 \cdot \binom{24}{2} = 1104$$

- One coordinate is +3 or −3 the other 23 are +1 or −1.

$$2^2 \cdot 24 = 98304$$

For instance, one from each group:

$$(2, -2, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, -2, -2, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0)$$

$$(0, 0, 0, 0, 0, 0, 0, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -4, 0, 0, 0, 0)$$

$$(1, -1, 1, 1, -1, 1, 1, 1, -1, 1, 1, 1, 3, -1, 1, 1, -1, 1, 1, -1, -1, -1, 1, -1)$$

If we calculate the distance of these points from the center (just by using the Euclidean distance, the square root of the sums of coordinate squares) we get $\sqrt{32}$ in all cases. This means that they are the same distance from the origin. Of course, by similar calculations we would still have to show that the spheres are separated and neighbouring ones touch each other.

The automorphism group of the Leech lattice is another sporadic group, $Co1$, discovered by John Horton Conway in 1968.

2.4.3 Moonshine Theory

Accidentally , in late 1970s John McKay found the number 196884 appearing in a number theoretical paper ([10] depicts the story in vivid details). This was indeed a totally unexpected connection between the monster group and modular functions. John Horton Conway named it "moonshine" with the meaning of nonexistent, foolish thing. The word also means illegally distilled whiskey – so one can see that mathematicians have a good sense of humour.

Later it turned out that Moonshine is not just a coincidence, but the theory has connections with physics, so it seems that somehow these giant algebraic structures are deeply engraved in our universe [6].

3 Summary

First thing we saw was that measuring can be considered generally and groups can measure the amount of symmetry an object has. Next we defined what is being simple for a symmetry group. Finally classifying the finite simple groups revealed some strange group structures and surprising connections with physics.

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