How to Talk to a Physicist: Groups, Symmetry, and Topology

Daniel T. Larson

July 6, 2005

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Chapter 1

Global Symmetries

1.1 Groups

We start with a set of elements G together with a rule (called multiplication) for combining two elements to get a third. The name of the resulting algebraic structure depends on the properties of the multiplication law. If the multiplication is associative, $(g_1g_2)g_3 = g_1(g_2g_3) = g_1g_2g_3$, then G is called a **semigroup**. If we also require an identity element $1 \in G$ such that $1 \cdot g = g = g \cdot 1 \ \forall g \in G$, then G becomes a **monoid**. Finally, the existence of inverse elements g^{-1} for every element such that $gg^{-1} = 1 = g^{-1}g \ \forall g \in G$ makes G into a **group**. The group multiplication law is not necessarily commutative, but if it is then the group is said to be **Abelian**. Because groups are closely related to symmetries, and symmetries are very useful in physics, groups have come to play an important role in modern physics.

Groups can be defined as purely abstract algebraic objects. For example, the group D_3 is generated by the two elements x and y with the relations $x^3 = 1$, $y^2 = 1$, and $yx = x^{-1}y$. We can systematically list all the elements: $D_3 = \{1, x, x^2, y, xy, x^2y\}$. We should check that this list is exhaustive, namely that all inverses and any combination of x and y appears in the list. Using the relations we see that $x^{-1} = x^2$ and $y^{-1} = y$, so $yx = x^{-1}y = x^2y$, all of which are already in the list. What about yx^2? The **order** of a group is the number of elements it contains and is sometimes written $|G|$. We see that D_3 is a group of order 6, i.e. $|D_3| = 6$. This is a specific example of a **dihedral group** D_n which is defined in general as the group generated by $\{x, y\}$ with the relations $x^n = 1$, $y^2 = 1$, and $yx = x^{-1}y$. We can systematically list the

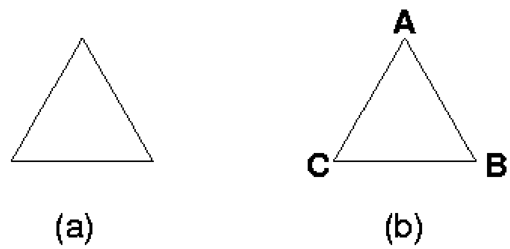

Figure 1.1: (a) An equilateral triangle; (b) the same triangle with labels.

elements: $D_n = \{1, x, x^2, \ldots, x^{n-1}, y, xy, x^2y, \ldots, x^{n-1}y\}$. You should prove that D_n has order $2n$.

Exercise 1 *The symmetric group S_n is the set of permutations of the integers $\{1, 2, \ldots, n\}$. Is S_n Abelian? Determine the order of S_n (feel free to start with simple cases of $n = 1, 2, 3, 4$). Note that $|S_3| = |D_3|$. Are they the same group (isomorphic)?*

1.1.1 Realizations and Representations

The previous example showed how a group can be defined in the abstract. However, groups often appear in the context of physical situations. Consider the rigid motions (preserving lengths and angles) in the plane that leave an equilateral triangle (shown in Figure 1.1(a)) unchanged. To keep track of what's going on, we will need to label the triangle as shown in Figure 1.1(b).

One rigid motion is a rotation about the center by an angle $\frac{2\pi}{3}$ in the counter-clockwise direction as shown in Figure 1.2(a). Let's call such a transformation R for "rotation". We say that R is a **symmetry** of the triangle because the triangle is unchanged after the action of R. Another symmetry is a reflection about the vertical axis, which we can call F for "flip". This is shown in Figure 1.2(b).

Both R and F can be inverted, by a clockwise rotation or another flip, respectively. Clearly combinations of Rs and Fs are also symmetries. For example, Figure 1.3(a) shows the combined transformation FR in the top panel, whereas the lower panel (b) shows the transformation R^2F. Note that our convention is that the right-most operation is done first.

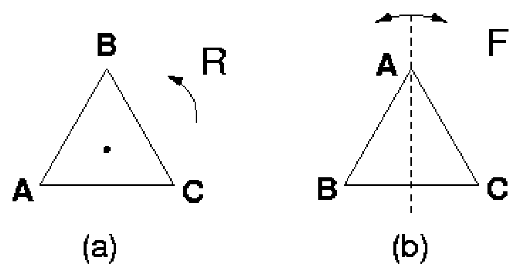

Figure 1.2: (a) R, a counter clockwise rotation by $\frac{2\pi}{3}$; (b) and the "flip" F, a reflection about the vertical axis (b).

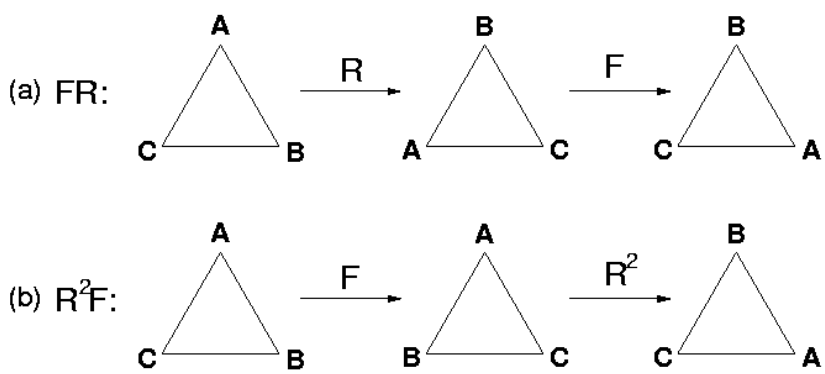

Figure 1.3: (a) The combined transformation FR; (b) the transformation R^2F.

These examples show that transformations on a space can naturally form a group. More formally, a group **realization** is a map from elements of G to transformations of a space M that is a **group homomorphism**, i.e. it preserves the group multiplication law. Thus if $T : G \to T(M) : g \mapsto T(g)$ where $T(g)$ is some transformation on M, then T is a group homomorphism if $T(g_1 g_2) = T(g_1)T(g_2)$. From this you can deduce that $T(1) = I$ where I is the identity ("do nothing") transformation on M, and $T(g^{-1}) = (T(g))^{-1} = T^{-1}(g)$.

Let's go back to the symmetries of our triangle. You probably noticed that $R^3 = I$ and $F^2 = I$, which bears striking similarities to D_3. In fact, with $T(x) = R$ and $T(y) = F$, it isn't hard to prove that T is a realization of D_3. One important thing to check is whether $T(yx) = T(y)T(x) = FR$ and $T(x^{-1}y) = T^{-1}(x)T(y) = R^{-1}F = R^2F$ are the same. But this is exactly what we showed in Figure 1.3.

In this example the map $T : D_3 \to \text{Symmetries}(\triangle)$ is bijective (one-to-one and onto) so in addition to being a homomorphism it is also an **isomorphism**. Such realizations are often called **faithful** because every different group element gets assigned to a different transformation. However, realizations do not need to be faithful. Consider the homomorphism $T' : D_3 \to \text{Symmetries}(\triangle)$ where $T'(x) = I$ and $T'(y) = F$. Then $T'(yx) = T'(y)T'(x) = FI = F$ and $T'(x^{-1}y) = T'(x^2y) = I^2F = F$. Thus the group relations and multiplication still hold so we have a realization, but it is definitely not an isomorphism.

Exercise 2 *Consider the map* $T'' : D_3 \to \text{Symmetries}(\triangle)$ *where* $T''(x) = R$ *and* $T''(y) = I$*. Is* T'' *is a realization?*

Exercise 3 *A* ***normal subgroup*** $H \subset G$ *is one where* $ghg^{-1} \in H$ *for all* $g \in G$ *and* $h \in H$*. Both* T' *and* T'' *map a different subgroup of* D_3 *to the identity. Are those normal subgroups? In this context, why is the concept of normal subgroup useful?*

Physicists are usually interested in the special class of realizations where M is a vector space and the $T(g)$ are linear transformations. Such realizations are called **representations**.[1] A **vector space** V is an Abelian group with

[1] A warning about terminology: Technically the representation is defined as the map (homomorphism) between G and transformations on a vector space. However, often the term "representation" is used to refer to the vector space on which the elements $T(g)$ act, and sometimes even to the linear transformations $T(g)$ themselves.

elements $|v\rangle$ called vectors and a group operation "+", and it possesses a second composition rule with scalars α which are elements of a field $\mathbb{F}$ (like $\mathbb{R}$ or $\mathbb{C}$) such that $\alpha|v\rangle \in V$ for all $\alpha \in \mathbb{F}$ and all $|v\rangle \in V$. Further we have the properties:

i) $(\alpha\beta)|v\rangle = \alpha(\beta|v\rangle)$

ii) $1 \in \mathbb{F}$ is an identity: $1|v\rangle = |v\rangle$

iii) $(\alpha+\beta)|v\rangle = \alpha|v\rangle + \beta|v\rangle$ and $\alpha(|v\rangle + |w\rangle) = \alpha|v\rangle + \alpha|w\rangle$.

A **linear transformation** on a vector space V is a map $T : V \to V$ such that $T(\alpha|v\rangle + \beta|w\rangle) = \alpha T(|v\rangle) + \beta T(|w\rangle)$. Linear algebra is essentially the study of vector spaces and linear transformations between them. This is important for us because soon we will see how quantum mechanics essentially boils down to linear algebra. Thus the study of symmetry groups in quantum mechanics becomes the study of group representations. But first we should look a some simple examples of representations.

Consider the parity group $P = \{x : x^2 = 1\}$, also known as $\mathbb{Z}_2$, the cyclic group of order 2. The most logical representation of P is by transformations on $\mathbb{R}$ where $T(x) = -1$. Clearly $T(x^2) = T(x)T(x) = (-1)^2 = 1$ so this forms a faithful representation. We could also study the **trivial representation** where $T'(x) = 1$. Again, $T'(x^2) = T'(x)^2 = 1^2 = 1$, so the group law is preserved, but nothing much happens with this representation, so it lives up to its name.

The **dimension** of a representation refers to the dimension of the vector space V on which the linear transformations $T(g)$ act, not to be confused with the order of the group. Let's now consider a two-dimensional representation of P. Take $\mathbb{R}^2$ with basis $|m\rangle, |n\rangle$ and let $T_2(x)|m\rangle = |n\rangle$ and $T_2(x)|n\rangle = |m\rangle$. You can check that $T_2(x^2) = (T_2(x))^2 = I$ because it takes $|m\rangle \to |n\rangle \to |m\rangle$ and $|n\rangle \to |m\rangle \to |n\rangle$. In terms of matrices we have:

$$T_2(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad T_2(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.1}$$

and you can check that $T_2(x)^2 = I$ in terms of matrices as well.

Something interesting happens if we change the basis of the vector space to $|\mu\rangle = \frac{1}{\sqrt{2}}(|m\rangle + |n\rangle)$ and $|\nu\rangle = \frac{1}{\sqrt{2}}(-|m\rangle + |n\rangle)$. Then

$$\begin{aligned}
T_2(x)|\mu\rangle &= \frac{1}{\sqrt{2}}(T_2(x)|m\rangle + T_2(x)|n\rangle) = \frac{1}{\sqrt{2}}(|n\rangle + |m\rangle) = |\mu\rangle \\
T_2(x)|\nu\rangle &= \frac{1}{\sqrt{2}}(-T_2(x)|m\rangle + T_2(x)|n\rangle) = \frac{1}{\sqrt{2}}(-|n\rangle + |m\rangle) = -|\nu\rangle
\end{aligned}$$

This new basis yields a new representation of P, call it T_2'. The matrices corresponding to the $|\mu\rangle, |\nu\rangle$ basis are:

$$T_2'(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad T_2'(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.2}$$

Of course, since T_2 and T_2' are related by a change of basis their matrices are related by a similarity transformation, $T_2(g) = ST_2'(g)S^{-1}$ and they aren't really different in any substantial way. Representations related by a change of basis are called **equivalent representations**.

Another thing to notice about T_2' is that all the representatives (i.e. both matrices) are diagonal. This means that the two basis vectors $|\mu\rangle$ and $|\nu\rangle$ are acted upon independently, so they each can be considered as separate one-dimensional representations. In fact, the representation on $|\mu\rangle$ is none other than the trivial representation, T', and the representation on $|\nu\rangle$ is the same as our faithful representation T above. A representation that can be separated into representations with smaller dimensions is said to be **reducible**. In general, a representation will be reducible when all of its matrices can be simultaneously put into the same block diagonal form:

$$T_{n+m+p}(g) = \left(\begin{array}{c|c|c} T_1(g) & & \\ \hline & T_2(g) & \\ \hline & & T_3(g) \end{array}\right) \begin{array}{l} \}_{n\times n} \\ \}_{m\times m} \\ \}_{p\times p} \end{array}. \tag{1.3}$$

Then each of the smaller subspaces are acted on by a single block and give a representation of the group G of lower dimension. Because larger representations can be built up out of smaller ones, it is sensible to try to classify the **irreducible** representations of a given group.

Exercise 4 *Can you construct an irreducible 2-dimensional representation of the parity group P? Can you construct a nontrivial 3-dimensional representation of P? If you can, is it irreducible?*

1.2 Lie Groups and Lie Algebras

Now we come to our main example, that of rotations. In $\mathbb{R}^3$ rotations about the origin preserve lengths and angles so they can be represented by 3×3 (real) orthogonal matrices. If we further specify that the determinant be equal to one (avoiding inversions) then we have the **special orthogonal group** called $SO(3)$:

$$SO(3) = \{\mathcal{O} \in 3 \times 3 \text{ matrices} : \mathcal{O}^\dagger = \mathcal{O}^{-1} \text{ and } \det \mathcal{O} = 1\}. \tag{1.4}$$

For real matrices like we have here, the Hermitian conjugate, $\mathcal{O}^\dagger = (\mathcal{O}^*)^T$ is equal to the transpose, $\mathcal{O}^T$, so I've chosen to use the former for later convenience. We can verify that this is a group by checking $(\mathcal{O}_1\mathcal{O}_2)^\dagger = \mathcal{O}_2^\dagger\mathcal{O}_1^\dagger = \mathcal{O}_2^{-1}\mathcal{O}_1^{-1} = (\mathcal{O}_1\mathcal{O}_2)^{-1}$ and $\det(\mathcal{O}_1\mathcal{O}_2) = \det\mathcal{O}_1 \det\mathcal{O}_2 = 1 \cdot 1 = 1$. Here our definition of the group is made in terms of a faithful representation. $SO(3)$ has two important features: it is a continuous group and it is *not* commutative. We will discuss these properties in turn.

Exercise 5 *Prove that orthogonal transformations on $\mathbb{R}^3$ do indeed preserve lengths and angles.*

Exercise 6 *Does the set of general $n \times n$ matrices form a group? If so, prove it. If not, can you add an additional condition to make it into a group.*

1.2.1 Continuous Groups

Rotations about the z-axis are elements (in fact, a subgroup) of $SO(3)$. Since we can imagine rotating by any angle between 0 and 2π it is clear that there are an infinite number of rotations about the z-axis and hence an infinite number of elements in $SO(3)$. We can write the matrix of such a rotation as

$$T_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.5}$$

demonstrating how such a rotation can be parameterized by a continuous real variable. It turns out that you need 3 continuous parameters to uniquely specify every element in $SO(3)$. (These three can be thought of as rotations about the 3 axes or the three Euler angles.) Because of the continuous parameters we have some notion of group elements being close together.

Such groups have the additional structure of a manifold and are called Lie groups.

A **manifold** is a set of point M together with a notion of open sets (which makes M a topological space) such that each point $p \in M$ is contained in an open set U that has a continuous bijection $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$. Thus an n-dimensional manifold is a space where every small region looks like $\mathbb{R}^n$. If all the functions φ are differentiable then you have a differentiable manifold.[2]

A **Lie group** is a group that is also a differentiable manifold.

Our main example of $SO(3)$ is in fact a Lie group. We already know how to deal with its group properties since matrix multiplication reproduces the group multiplication law. But what kind of manifold is $SO(3)$?

We will begin to answer this question algebraically. More generally, $SO(n)$ consists of $n \times n$ real matrices with $\mathcal{O}^\dagger\mathcal{O} = I$ and $\det \mathcal{O} = 1$. A general $n \times n$ real matrix has n^2 entries so is determined by n^2 real parameters. But the orthogonality condition gives $\frac{n(n+1)}{2}$ constraints (because the condition is symmetric, so constraining the upper triangle of the matrix automatically fixes the lower triangle). Since any orthogonal matrix must have $\det \mathcal{O} = \pm 1$, the constraint for a positive determinant only eliminates half of the possibilities but doesn't reduce the number of continuous parameters. Thus a matrix in $SO(n)$ will be specified by $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$. Thus $SO(3)$ is specified by 3 parameters and is therefore a 3-dimensional manifold.

Knowing the dimension is a start, but we can learn more by using geometric reasoning. Let's specify a rotation about an axis by a vector along that axis with length in the range $[0, \pi]$ corresponding to the counter-clockwise angle of rotation about that axis. The collection of all such vectors is a solid, closed ball of radius π in $\mathbb{R}^3$, call it D^3 (for "disk). However, a rotation by π about some axis $\vec{n}$ is the same as a rotation by π about $-\vec{n}$. So to take this into account we need to specify that opposite points on the surface of ball are actually the same. If this identification is made into a formal equivalence relation $\sim$ then we have $SO(3) \cong D^3/\sim$. So as a manifold $SO(3)$ can be visualized as a three-dimensional solid ball with opposite points on the surface of the ball identified. As a preview of what is to come, note that this shows that $SO(3)$ is *not* simply connected.

Exercise 7 *What is the relationship between* $SO(3)$ *and the 3-dimensional*

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[2]For a general abstract manifold the definition of differentiable is that for two overlapping regions U_i and U_j with corresponding maps φ_i and φ_j the composition $\varphi_i \circ \varphi_j^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is infinitely differentiable.

unit sphere, S^3?

Exercise 8 *Does the set of general $n \times n$ matrices form a manifold? If so, what is its dimension? If not, can you add an additional condition to make it into a manifold?*