Representations of the Poincaré group

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Abstract

In this note, we will discuss representations of the Poincaré group. I later also discuss how one gauges a global symmetry.

1 Symmetry and Eigenstates of the Hamiltonian

In QM, we are interested in the eigenstates of the Hamiltonian, H satisfying

$$H|n\rangle = E_n |n\rangle \tag{1}$$

These states could in general be degenerate with degeneracy g_n. A general eigenstate of the Hamiltonian can then be written as $|n,i\rangle$ where $i = 1, \cdots, g_n$ labels the basis vectors of the subspace defined by vectors with eigenvalue E_n. These eigenstates span the entire Hilbert space, which can be written as

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \tag{2}$$

where $\mathcal{H}_n$ is defined as the subspace with eigenvalue E_n. The dimension of $\mathcal{H}_n$ is g_n.

A symmetry transformation is defined as a transformation that does not change any correlation functions (and the norms) of the quantum theory. Consider a transformation on the eigenstates $|n,i\rangle \to G|n,i\rangle$. Under this transformation, the norm becomes

$$\langle n,i | n,i\rangle \to \langle n,i| G^\dagger G |n,i\rangle \tag{3}$$

Invariance of the norm implies $G^\dagger G = 1$, i.e. G is a unitary operator. The eigenvalue of the state under this transformation becomes

$$\langle n,i| H |n,i\rangle \to \langle n,i| G^\dagger H G |n,i\rangle \tag{4}$$

Invariance of the eigenvalue implies $G^\dagger H G = H \implies [H,G] = 0$. Given these conditions for a symmetry transformation, consider the eigenvalue of the state $G|n,i\rangle$. This is

$$H(G|n,i\rangle) = G(H|n,i\rangle) = E_n(G|n,i\rangle) \tag{5}$$

Thus, $G|n,i\rangle$ has the eigenvalue E_n. Thus, we must have

$$G|n,i\rangle = \sum_{i=1}^{g_n} c_i |n,i\rangle \tag{6}$$

In other words, the states $|n,i\rangle$ transform under the g_n-dimensional representation of the symmetry group G. Now, any finite dimensional group can be written as a direct sum of irreducible representations. We can summarize this entire discussion in a theorem

Theorem: If the Hamiltonian commutes with all the elements of a representation of a group G, then we can choose the eigenstates of H to transform according to irreducible representations of G.

What does this imply for QFT? We know that any Hamiltonian we write down, will be invariant under Poincaré transformations. Thus, the eigenstates of the Hamiltonian can be chosen to transform under irreducible representations of the Poincaré group. To construct the Hilbert space, we must then learn the representations of the Poincaré group.

2 Representations of the Poincare group

For the discussion in the previous section, we realise that we need to write down irreducible representations of the Poincaré group. To see how one would go about doing this, let us quickly go through the process for $SU(2)$ which you should already be familiar with.

2.1 Representations of $SU(2)$

Recall in QM, the angular momentum satisfied the algebra of $SU(2)$

$$[L_i, L_j] = i\epsilon_{ijk}L_k \tag{7}$$

To define representations of this algebra, we first look for operators that commute with all the generators. In this case, we find that $[L^2, L_i] = 0,\ \forall i$. The representations of $SU(2)$ were then labelled by the eigenvalue of L^2, the *spin*. Thus, the Hilbert space could be defined as

$$\mathcal{H} = \bigoplus_s |s\rangle \tag{8}$$

We then proceed to write down the explicit form of the spin s representation. To do this, we first *fix* s. For a fixed s, we look for an operator that commutes with L^2. By convention we choose L_z. We define the states in the this representation to be eigenstates of L_z as well and label them as $|s, m\rangle$. Then we define the ladder operators $L_\pm = L_1 \pm iL_2$. From the algebra of the group, we can easily verify

$$\begin{aligned} J_+ |s, m\rangle &= \sqrt{(s-m)(s+m+1)}\,|s, m+1\rangle \\ J_- |s, m\rangle &= \sqrt{(s+m)(s-m+1)}\,|s, m-1\rangle \end{aligned} \tag{9}$$

This then implies that $-s \le m \le s$. Further since $J_\pm$ raises and lowers the L_z by an integer, we must have $2s \in \mathbb{Z} \implies s \in \frac{\mathbb{Z}}{2}$. This implied that the spin s representation of $SU(2)$ is $(2s+1)$-dimensional. (and in particular, finite) We are done! The Hilbert space is then written as

$$\mathcal{H} = |0\rangle \oplus \left|\frac{1}{2}, m_{\frac{1}{2}}\right\rangle \oplus |1, m_1\rangle \oplus \cdots \tag{10}$$

where $s \le m_s \le s$. This procedure is important and we summarize it again.

- We search for operators that commuted with all generators in the algebra (L^2). The representation thus constructed was labelled by the eigenvalue of this operator s.[1]
- Fix an s, and then choose a second operator (L_z) to construct the basis of the spin s representation.
- Construct raising and lowering operators, and hence construct the full representation.

The same prescription above will be used to construct representations of the Poincaré group.

[1] Operators that commute with all generators in the algebra are called Casimir operators

2.2 Representations of the Poincaré group

Let us follow the above prescription and apply it to the Poincaré group. The Poincaré algebra reads

$$\begin{aligned}
[J^{\mu\nu}, J^{\rho\sigma}] &= i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right) \\
[P^\mu, P^\nu] &= 0 \\
[P^\mu, J^{\rho\sigma}] &= i\left(g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho\right)
\end{aligned} \tag{11}$$

Let us first find all the Casimir Invariants of the Poincaré algebra. There are two Casimir invariants of the Poincaré group. These are $P^2 = P^\mu P_\mu$ and $W^\mu W_\mu$, where W^μ is the Pauli-Lubanski pseudovector,

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma \tag{12}$$

It is easy to check that these satisfy

$$[W^\mu, P^\nu] = 0,\ [W^\mu, J^{\alpha\beta}] = 0 \implies [W^2, P^\mu] = [W^\mu, J^{\mu\nu}] = 0 \tag{13}$$

Moreover, they also satisfy $P^\mu W_\mu = 0$ (This is what gives rise to the concept of *little group* as we will see).

Now since P^2 is a Casimir invariant of the group, the representations of the Poincaré group are labelled by its eigenvalue, m^2. Let us now fix $P^2 = m^2$ and look at the structure of the representation labelled by m.

- Massive representations ($m^2 > 0$): These are representations with $(P^0)^2 - (P^i)^2 = m^2$. However, there are infinite number of eigenvalues of P^0 and P^i that satisfy $P^2 = m^2$, the mass m representation of the Poincaré group must be infinite dimensional! What do we do next? The prescription tells us that we should FIX a P^μ and then look at the second Casimir invariant to fix the representations. Without loss of generality let us fix $P^\mu = (m, 0, 0, 0)$ (for a different P, we can always do a Lorentz transformation to bring into this form). The Pauli-Lubanski pseudovector then satisfies $W^\mu P_\mu = 0 \implies W^0 = 0$. Now, since W^μ commutes with all the generators of the Poincaré group, the W^i must be invariant under the Poincaré group as well. However, the only transformations that we can do that do not change P^μ are rotations, generated by L^i. Therefore, the remaining W^i must have the $SO(3) \equiv SU(2)$ algebra. This is the whole concept of the *little group*. Let us see this explicitly. When $P^\mu = (m, 0, 0, 0)$, we have

$$W^\mu = -\frac{1}{2}m\epsilon^{\mu\nu\rho 0}J_{\nu\rho} \implies W^0 = 0, \quad W^i = \frac{m}{2}\epsilon^{0ijk}J_{jk} = -mL^i \tag{14}$$

  but L^i exactly satisfies the algebra of $SU(2)$. The representations must then be labelled by spin s (which are eigenvalues of the operator $W^2 = -(W^i)^2$, and must be $(2s+1)$-dimensional. Therefore, *Massive representations of the Poincaré group are labelled by the mass m and spin s. Representations with fixed P^μ are $(2s+1)$-dimensional.*

- Massless representations: These are representations with $P^2 = 0$. Again, since there are an infinite number of P^μ that can satisfy this, the representation is infinite dimensional. Without loss of generality, we can write $P^\mu = (E, 0, 0, E)$. The Pauli-Lubanski pseudovector becomes

$$W^\mu = \frac{E}{2}\epsilon^{0\mu jk}J_{jk} - \frac{E}{2}\epsilon^{\mu\nu\rho 3}J_{\nu\rho} \tag{15}$$

  Thus

$$\begin{aligned}
W^0 &= -E\hat{P}\cdot\vec{L} \\
W^1 &= -EL^1 - EK^2 \\
W^2 &= -EL^2 + EK^1 \\
W^3 &= -E\hat{P}\cdot\vec{L}
\end{aligned} \tag{16}$$

These have the algebra

$$
\begin{aligned}
[W^1, W^2] &= E^2[L^1 + K^2, L^2 - K^1] = 0 \\
[W^2, W^3] &= E^2[L^2 - K^1, L^3] = -iEW^1 \\
[W^3, W^1] &= E^2[L^3, L^1 + K^2] = -iEW^2
\end{aligned}
\tag{17}
$$

This is precisely the algebra of $SE(2)$ or $ISO(2)$. This is what we mean when we say that the *little group* of the Poincaré group for massless case is $ISO(2)$.

The representation is then labelled by the eigenvalues of W^2. However, for this case, it is easy to see that the eigenvalues of W^2 are all zero. This is therefore, not a good label. We do the next best thing. We label the representations by the eigenvalues of W^i. It is immediately easy to see that the eigenvalues of W^1 and W^2 are also identically zero! Therefore, the representation is only labelled by the eigenvalues of $W^3 = \hat{P} \cdot \vec{L}$. But this is just the helicity operator! Therefore, *Massless representations of the Poincaré group are labelled by their helicity h*. However, these representations are 1-dimensional. Therefore, massless representations of the Poincaré group are 1-dimensional. However, note that W^μ is a pseudovector. Under parity transformations the eigenvalue $h \to -h$. If we wish to describe particles that preserve parity, we must put the h and $-h$ representations together to describe the particle (since parity would take $h \to -h$). Thus, massless representations of the full Poincaré group are 2-dimensional, for all helicities $h > 0$. The only special case is when $h = 0$, which defines a massless scalar, which has only 1 degree of freedom.