Wigner-İnönü Contraction

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Nature and Nature's laws lay hid in night:
God said, "Let Newton be!" and all was light. — Alexander Pope

I. INTRODUCTION

When it comes to the relation between Lorentz group and Galilean group, one can't stop but to nag about the limit $c \to \infty$ would "contract" (whatever it means) the former to the latter. Now let's take a quick glance upon the Lorentz algebra:

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk} J_k, \\ [J_i, K_j] &= i\epsilon_{ijk} K_k, \\ [K_i, K_j] &= -i\epsilon_{ijk} J_k, \end{aligned} \tag{1}$$

and the Galilean group:

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk} J_k, \\ [J_i, K_j] &= i\epsilon_{ijk} K_k, \\ [K_i, K_j] &= 0. \end{aligned} \tag{2}$$

Wait, where is c? There is no such thing as *light* in them! Then what do we mean by taking the limit $c \to \infty$?

Admittedly, there is some technical problems in saying "taking the limit $c \to \infty$ of the group". There is no c in the algebra nor the group, but only when one talks about the *representation* does the light come out.

For example, consider a (1+1) dimensional Lorentz group realized on the vector space (x, ct):

$$\Lambda(v) = \begin{pmatrix} \gamma & \gamma \frac{v}{c} \\ \gamma \frac{v}{c} & \gamma \end{pmatrix}, \tag{3}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$. Aha! Here is the c we've been looking for! However, life is never easy. If we naively take the $c \to \infty$ limit, then

$$\Lambda_\infty(v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4}$$

Oops, it's a catastrophe: the representation becomes trivial. It's not the faithful Galilean group representation we expected. Fortunately, we have a way to save it. Apply a c-dependent similar transformation

$$C = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \tag{5}$$

on $\Lambda(v)$ and *then* take the $c \to \infty$ limit:

$$C\lambda(v)C^{-1} \to \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}. \tag{6}$$

Bingo! This is the faithful representation of Galilean group.

The above example makes it clear that there is always some ambiguities in taking the contraction limit of a representation of a group. It is then desirable to develop a systematic and representation-independent procedure for the contraction. This is the **Wigner-İnönü Contraction**[1][2].

II. FORMAL DEVELOPMENT

Consider an algebra $\{X_i\}$ with the commutator:

$$[X_i, X_j] = i c_{ij}{}^k X_k. \tag{7}$$

If one subjects the X_i to a nonsingular transformation U, one obtains new operators Y_i:

$$Y_i = U_i{}^j X_j, \tag{8}$$

which still generate the original group but with the new structure constant given by

$$C_{ij}{}^k = U_i{}^m U_j{}^n c_{mn}^{\ \ l} \left(U^{-1}\right)_l{}^k. \tag{9}$$

What if U *is* singular? Then the group generated by Y_i will no longer be the same. Consider a U which depends on ϵ in the following form:

$$U = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}, \tag{10}$$

where 1 and ϵ are understood as $r \times r$ and $(n-r) \times (n-r)$ matrices, respectively. Apparently, U becomes singular when $\epsilon = 0$, and it is this limit we want to exploit, which we call "contraction".

Before moving on, it is useful to divide two kinds of indices, 1μ and 2λ, which stands for the first r indices and the remaining part. (1 and 2 are not additional indices; they are here to remind you that they are of different divisions.) With this specific form for U, Eq.[8] becomes

$$\begin{aligned} &Y_{1\mu} = X_{1\mu},\ \mu = 1 \sim r, \\ &Y_{2\lambda} = \epsilon X_{2\lambda},\ \lambda = r+1 \sim n. \end{aligned} \tag{11}$$

The contraction limit doesn't necessarily exist for any kinds of $c_{ij}{}^k$. In fact, from the analysis in the reference, the sufficient and necessary condition for the contraction to exist is

$$c_{1\mu 1\nu}^{\ \ 2\lambda} = 0, \tag{12}$$

i.e., $X_{1\mu}$'s form a subalgroup, called H. Furthermore, the new structure constants from the transformation Eq.[10] given by Eq.[9], together with the requirement Eq.[12], read

$$C_{1\mu 1\nu}^{\ \ 1\kappa} = c_{1\mu 1\nu}^{\ \ 1\kappa},\quad C_{1\mu 1\nu}^{\ \ 2\lambda} = c_{1\mu 1\nu}^{\ \ 2\lambda} = 0, \tag{13}$$

$$C_{1\mu 2\lambda}^{\ \ 1\kappa} = 0,\quad C_{1\mu 2\sigma}^{\ \ 2\lambda} = c_{1\mu 2\sigma}^{\ \ 2\lambda}, \tag{14}$$

$$C_{2\lambda 2\sigma}^{\ \ 1\mu} = C_{2\lambda 2\sigma}^{\ \ 2\delta} = 0. \tag{15}$$

Eq.[13] says $Y_{1\mu}$'s generate an invariant subgroup, while Eq.[14] and Eq.[15] say that $Y_{2\lambda}$'s form an abelian invariant subgroup. The result can be summarized as two theorems:

Theorem1 Given a Lie group G, the contraction could take place if and only if there exists a nontrivial subgroup H. The algebra for H remains fixed under contraction, while the remaining contracted algebra generates an abelian invariant subgroup, called N, of the contracted group G'. Furthermore, G' is the semidirect product of N and H, or equivalently, $H \simeq G'/N$.

Theorem2 Conversely, the necessary condition for a group G' to be derivable from another group by contraction is the existence in G' of an abelian invariant subgroup N and subgroup H such that G' is the semidirect product of them.

These are the main theorems in this note. The proof is omitted. It indicates the close connection between the semidirect product and the group contraction. Let's see some examples below.

III. EXAMPLES

Example 1 Consider the **SO(3)** group algebra :

$$[X_1, X_2] = iX_3, \tag{16}$$
$$[X_2, X_3] = iX_1, \tag{17}$$
$$[X_3, X_1] = iX_2. \tag{18}$$

Now take the SO(2) group generated by X_3 as the invariant subgroup H. If one applies the ϵ-dependent similar transformation U in Eq.[10], the new generators Y_i's are

$$Y_1 = \epsilon X_1, \tag{19}$$
$$Y_2 = \epsilon X_2, \tag{20}$$
$$Y_3 = X_3. \tag{21}$$

Take the contraction limit $\epsilon \to 0$, the new algebra for Y_i's then reads

$$[Y_1, Y_2] = \epsilon^2[X_1, X_2] = i\epsilon^2 Y_3 \to 0, \tag{22}$$
$$[Y_2, Y_3] = \epsilon[X_2, X_3] = iY_1, \tag{23}$$
$$[Y_3, Y_1] = \epsilon[X_3, X_1] = iY_2. \tag{24}$$

This is just the algebra for ISO(3), and the abelian invariant subgroup N is the two dimensional translation group generated by Y_1 and Y_2. Conclusion: *SO(3) is contracted to ISO(2).*

This example has a simple physical explanation: At first the symmetry at hand is SO(3), but if one restricts two kinds of rotation, X_1 and X_2, only to their infinitesimal version, Y_1 and Y_2, then they "look like" translations in the contraction limit. Together with the unaltered generator $X_3 = Y_3$, Y_i's form a ISO(2) group.

Example 2 Recall the original problem we tackled with: the **Lorentz group** and the **Galilean group**, Eq.[1] and Eq.[2]. Now the invariant subgroup H is taken to be the SO(3) rotation group, generated by J_i, and the boost generators are contracted. Namely,

$$J_i' = J_i, \tag{25}$$
$$K_i' = \epsilon K_i \equiv \frac{1}{c} K_i. \tag{26}$$

Now, put the new generators into the commutators and take the contraction limit $c \to \infty$, we obtain the Galilean algebra, Eq.[2]. Note how the c comes out: it is the contraction parameter that controls the singular nature of U. Apparently, this procedure is representation-independent, and we are free from the ambiguity when tackling with the contraction of representation.

Example 3 As a final example, consider the **de Sitter group SO(4,1)**:

$$[J^{AB}, J^{CD}] = i\left(\eta^{AC} J^{BD} - \eta^{BC} J^{AD} - (C \leftrightarrow D)\right), \; A, B, C, D = 0 \sim 4, \tag{27}$$

where $\eta^{AB} = diag(-1, 1, 1, 1, 1)$. If we separate the $J^{4\mu}$'s generators, the commutators become

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(\eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\rho} J^{\mu\sigma} - (\rho \leftrightarrow \sigma)\right), \tag{28}$$
$$[J^{4\mu}, J^{\rho\sigma}] = i\left(\eta^{\mu\sigma} J^{4\rho} - \eta^{\mu\rho} J^{4\sigma}\right), \tag{29}$$
$$[J^{4\mu}, J^{4\nu}] = iJ^{\mu\nu}, \qquad \mu, \nu, \rho, \sigma = 0 \sim 3. \tag{30}$$

Then apply the transformation

$$P^\mu = \epsilon J^{4\mu} \equiv \frac{1}{L} J^{4\mu}, \tag{31}$$

where L is the contraction parameter, the size of the universe. Taking the contraction limit $L \to \infty$, de Sitter group becomes the **Poincaré group**:

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(\eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\rho} J^{\mu\sigma} - (\rho \leftrightarrow \sigma)\right), \tag{32}$$
$$[P^\mu, J^{\rho\sigma}] = i\left(\eta^{\mu\sigma} P^\rho - \eta^{\mu\rho} P^\sigma\right), \tag{33}$$
$$[P^\mu, P^\nu] = 0. \tag{34}$$

Note that the original nonabelian invariant subalgebra Eq.[30] becomes abelian invariant subalgebra Eq.[34] under contraction. Physically, this means that any infinitesimal group transformation commutes with each other.

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[1] E. İnönü, E. P. Wigner, Proc. Nat. Acad. Sci.,39, 510-524 (1953).
[2] F. Gursey, Group Theoretical Concepts and Methods in Elementary Particle Physics.