9 Quantum Field Theory for Children

The theories (known and hypothetical) needed to describe the (very) early universe are *quantum field theories* (QFT). The fundamental entities of these theories are *fields*, i.e., functions of space and time. For each particle species, there is a corresponding field, having at least as many (real) components φ_i as the particle has internal degrees of freedom. For example, for the photon, the corresponding field is the vector field $A^\mu = (A^0, A^1, A^2, A^3) = (\phi, \vec{A})$, the 4-vector potential, already familiar from electrodynamics. The photon has two internal degrees of freedom. The larger number of components in A^μ is related to the *gauge freedom* of electrodynamics.

In classical field theory the evolution of the field is governed by the *field equation*. Quantizing a field theory gives a quantum field theory. *Particles are quanta of the oscillations of the field around the minimum of its potential.* The field value at the potential minimum is called the *vacuum*. Up to now, we have described the events in the early universe in terms of the *particle picture*. However, the particle picture is not fundamental, and can be used only when the fields are doing small oscillations. For many possible events and objects in the early universe (inflation, topological defects, spontaneous symmetry breaking phase transitions) the field behavior is different, and we need to describe them in terms of field theory. In some of these topics classical field theory is already sufficient for a reasonable and useful description.

9.1 Zero-Temperature Field Theory

In this section we discuss "zero-temperature" field theory in Minkowski space, i.e., we forget high-temperature effects and the curvature of spacetime.

The starting point in field theory is the *Lagrangian density* $\mathcal{L}(\varphi_i, \partial^\mu\varphi_i)$. The simplest case is the scalar field φ, for which

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - V(\varphi)\,. \tag{1}$$

Here $V(\varphi)$ is the *potential* of the field. If

$$V(\varphi) = \frac{1}{2}m^2\varphi^2\,, \tag{2}$$

the particle corresponding to the field φ will have mass m. In general, the mass of the particle is given by $m^2 = V''(\varphi)$. We write

$$V'(\varphi) \equiv \frac{dV}{d\varphi} \qquad \text{and} \qquad V''(\varphi) \equiv \frac{d^2V}{d\varphi^2}\,. \tag{3}$$

The particles corresponding to scalar fields are spin-0 bosons. Spin-$\frac{1}{2}$ particles correspond to spinor fields and spin-1 particles to vector fields.

The *field equation* is obtained from the Lagrangian by minimizing (or extremizing) the action

$$\int \mathcal{L}d^4x\,, \tag{4}$$

which leads to the *Euler–Lagrange equation*

$$\frac{\partial\mathcal{L}}{\partial\varphi_i(x)} - \partial_\mu\frac{\partial\mathcal{L}}{\partial[\partial_\mu\varphi_i(x)]} = 0\,. \tag{5}$$

For the above scalar field we get the field equation

$$\partial_\mu \partial^\mu \varphi - V'(\varphi) = 0\,. \tag{6}$$

For a massless noninteracting field, $V(\varphi) = 0$, and the field equation is just the wave equation

$$\partial_\mu \partial^\mu \varphi = -\ddot{\varphi} + \nabla^2 \varphi = 0\,. \tag{7}$$

The Lagrangian also gives us the energy tensor

$$T^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial^\nu \varphi + g^{\mu\nu} \mathcal{L}\,. \tag{8}$$

For the scalar field

$$T^{\mu\nu} = \partial^\mu \varphi \partial^\nu \varphi - g^{\mu\nu} \big[\frac{1}{2} \partial_\rho \varphi \partial^\rho \varphi + V(\varphi)\big]\,. \tag{9}$$

In particular, the energy density and pressure of a scalar field are

$$\rho \;=\; T^{00} \;=\; \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}\nabla\varphi^2 + V(\varphi) \tag{10}$$

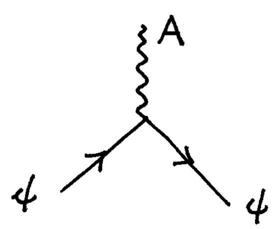
$$p \;=\; \frac{1}{3}\left(T^{11} + T^{22} + T^{33}\right) \;=\; \frac{1}{2}\dot{\varphi}^2 - \frac{1}{6}\nabla\varphi^2 - V(\varphi)\,. \tag{11}$$

(We are in Minkowski space, so that $g^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$).

Interactions between particles of two different species are due to terms in the Lagrangian which involve both fields. For example, in the Lagrangian of quantum electrodynamics (QED) the term

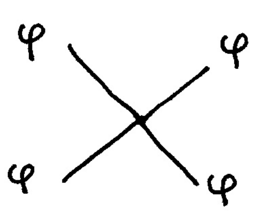
$$-ie\psi^\dagger \gamma^0 \gamma^\mu A_\mu \psi \tag{12}$$

is responsible for the interaction between photons (A^μ) and electrons (ψ). (The γ^μ are Dirac matrices). A graphical representation of this interaction is the Feynman diagram



A higher power (third or fourth) of a field, e.g.,

$$V(\varphi) = \frac{1}{4}\lambda\varphi^4\,, \tag{13}$$



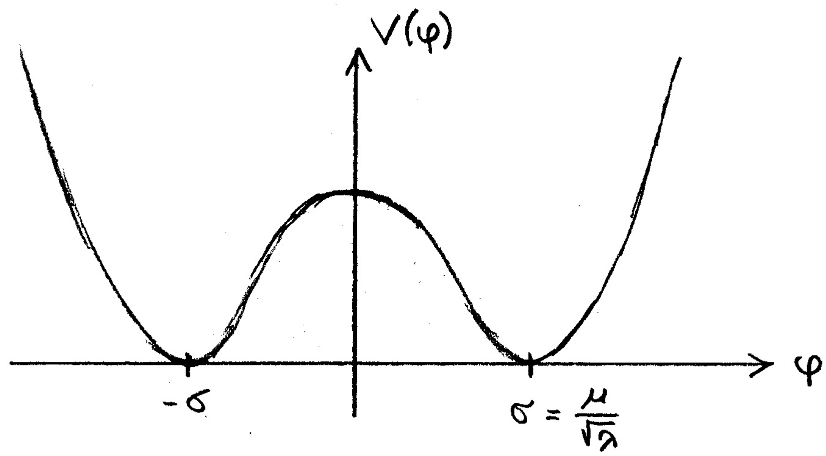


Figure 1: Potential giving rise to spontaneous symmetry breaking.

represents self-interaction. In QCD, gluons have this property.

Some theories exhibit *spontaneous symmetry breaking* (SSB). For example, the potential

$$V(\varphi) = V_0 - \frac{1}{2}\mu^2\varphi^2 + \frac{1}{4}\lambda\varphi^4 \tag{14}$$

has two minima, at $\varphi = \pm\sigma$, where $\sigma = \mu/\sqrt{\lambda}$. At low temperatures, the field is doing small oscillations around one of these two minima (see Fig. 1). Thus the vacuum value of the field is nonzero. If the Lagrangian has interaction terms, $c\varphi\psi^2$, with other fields ψ, these can now be separated into a mass term, $c\sigma\psi^2$ and an interaction term, by redefining the field φ as

$$\varphi = \sigma + \tilde{\varphi} \;\Rightarrow\; c\varphi\psi^2 = c\sigma\psi^2 + c\tilde{\varphi}\psi^2\,. \tag{15}$$

Thus spontaneous symmetry breaking gives the ψ particles a mass $\sqrt{2c\sigma}$. This kind of a field φ is called a *Higgs field*. In electroweak theory the fermion masses are due to a Higgs field.

9.2 High Temperature QFT

The material in subsections 9.2 and 9.3 is not needed in the rest of the course. They were meant as preparation for topics (QCD phase transition, electroweak phase transition, grand unified theories, topogical defects), that have now been dropped from the course. Thus you can skip these sections.

When the temperature is comparable to (or larger than) the energy scale of the theory[1], thermodynamic effects become important. The values around which the fields fluctuate are no more those which minimize the energy (the minimum of the potential $V(\varphi)$), but rather those which minimize the free energy (the thermodynamic potential).

[1]The energy scale is given by the constants in the Lagrangian, such as the m, μ, and λ in Eqs. (2),(13), and (14).

From thermodynamics we have the relation

$$F = E - TS \tag{16}$$

relating the free energy F, the energy E, the temperature T, and the entropy

$$S \equiv -\left(\frac{\partial F}{\partial T}\right). \tag{17}$$

In thermal QFT the thermodynamic variables are the field values $\varphi = \langle\varphi\rangle$ (their expectation values at local thermal equilibrium) and the temperature T. The free energy density is called the *effective potential* $V(\varphi, T)$. It is related to the energy density ρ by

$$V(\varphi, T) = \rho(\varphi, T) - Ts(\varphi, T), \tag{18}$$

where

$$s(\varphi, T) \equiv -\frac{\partial V(\varphi, T)}{\partial T} \tag{19}$$

is the entropy density.

We shall not discuss how the effective potential is calculated. For example, for the symmetry breaking scalar field whose classical potential is

$$V(\varphi) = V_0 - \frac{1}{2}\mu^2\varphi^2 + \frac{1}{4}\lambda\varphi^4\,, \tag{20}$$

the effective potential is

$$V(\varphi, T) = V(\varphi) + \frac{1}{24}m^2(\varphi)T^2 - \frac{\pi^2}{90}T^4 + \text{quantum corrections}\,, \tag{21}$$

when $T \gg m$. Here

$$m^2(\varphi) = V''(\varphi) = -\mu^2 + 3\lambda\varphi^2\,. \tag{22}$$

For the energy density of this scalar field we thus get

$$\begin{aligned} \rho(\varphi, T) &= V(\varphi, T) - T\frac{\partial V(\varphi, T)}{\partial T} \\ &= V(\varphi) - \frac{1}{24}m^2(\varphi)T^2 + \frac{\pi^2}{30}T^4 + \text{q.c.} \end{aligned} \tag{23}$$

We recognize the term $(\pi^2/30)T^4$, the energy density of spin-0 bosons at $T \gg m$. Comparing to the $T = 0$ energy density of a classical scalar field,

$$\rho = V(\varphi) + \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}\nabla\varphi^2\,, \tag{24}$$

we notice that the gradient terms $\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}\nabla\varphi^2$ are missing. This is because now φ represents the expectation value of the field, which is assumed homogeneous because we are describing a system in thermal equilibrium. The contribution to the energy from the fluctuations of the field around this equilibrium value is now represented in a statistical manner by the temperature-dependent terms instead of the gradient terms which assume a particular field configuration.

Often it makes sense to separate from each other the slow, large-scale, “classical” behavior of the field φ, and the microscopic fluctuations of the field, over which “we

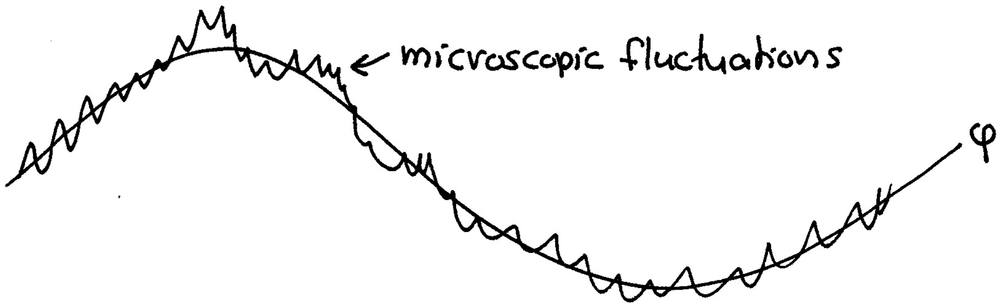


Figure 2: Separating out the large and small scales.

have integrated" and whose average effect is represented by the effective potential $V(\varphi, T)$. See Fig. 2. Then, in this large-scale view, T and φ may be inhomogeneous and time-dependent, but locally we assume thermal equilibrium, represented by the local values of T and φ. Then the large-scale behavior of the field is described by the field equation

$$\partial_\mu \partial^\mu \varphi - \frac{\partial V(\varphi, T)}{\partial \varphi} = 0\,, \tag{25}$$

where the potential $V(\varphi)$ has now been replaced by the effective potential. This equation includes the small-scale quantum and thermal effects only in an average sense, and does not therefore describe random events possibly caused by these fluctuations.

9.3 Phase transitions

Let us now consider a system with spontaneous symmetry breaking. When $T \to 0$, $V(\varphi, T) \to V(\varphi)$, and the system settles into $\varphi = \pm\sigma$. As the temperature becomes higher, the shape of $V(\varphi, T)$ as a function of φ changes, and at a sufficiently high temperature, $T > T_c$, the minimum of the effective potential is at $\varphi = 0$, and the system settles into a symmetric state $\varphi = 0$. We call this a *phase transition* from the *broken* phase to the *symmetric* phase.

In the early universe, the temperature was high, and the universe was in the symmetric phase. As the temperature fell below the *critical temperature* T_c the universe underwent a phase transition to the broken phase.

Such a phase transition can be either *first* or *second* order.

In a first-order phase transition the effective potential has two (local) minima at a temperature range (T_-, T_+), the *true vacuum* or the global minimum and the *false vacuum*. See Fig. 3. At the critical temperature T_c, $(T_- < T_c < T_+)$, the effective potential has the same value in both minima. When the temperature falls below T_c the field φ would like to be in the minimum corresponding to the broken phase, $\varphi = \varphi_b(T)$, but it has to remain some time in the symmetric phase, $\varphi = 0$, because there is a potential barrier separating the two minima. The state $\varphi = 0$ is now *metastable*. We say that the system is *supercooled*. The field gets over the barrier by thermal fluctuation or through it by quantum tunneling, which are random events occurring in different places at different times. When this happens at some small

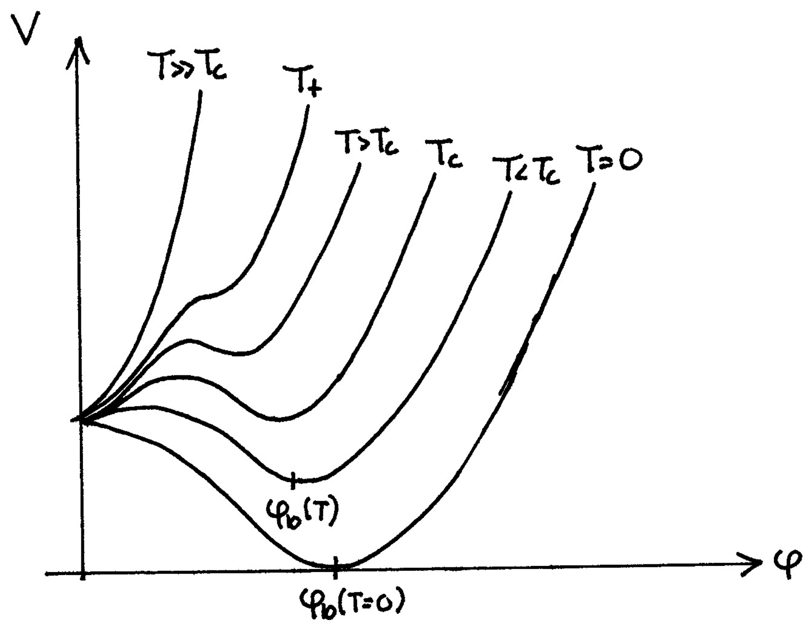


Figure 3: The effective potential for a first-order phase transition.

region, we say that a *bubble* of the broken phase has nucleated. A bubble is a field configuration where the field is at $\varphi = \varphi_b(T)$ at center and at $\varphi = 0$ further out. See Fig. 4. After the nucleation the field evolution can again be described by the field equation. At the phase boundary (the *bubble wall*) the field moves from the symmetric phase to the broken phase and the bubble grows. As old bubbles grow and new bubbles are nucleated, the broken phase gradually takes over.

In a first-order phase transition it takes a significant amount of time to convert the whole universe from the old phase to the new phase, because the expansion of the universe has to make space for the *latent heat*

$$L \equiv \rho_\mathrm{s}(T_\mathrm{c}) - \rho_\mathrm{b}(T_\mathrm{c}) \tag{26}$$

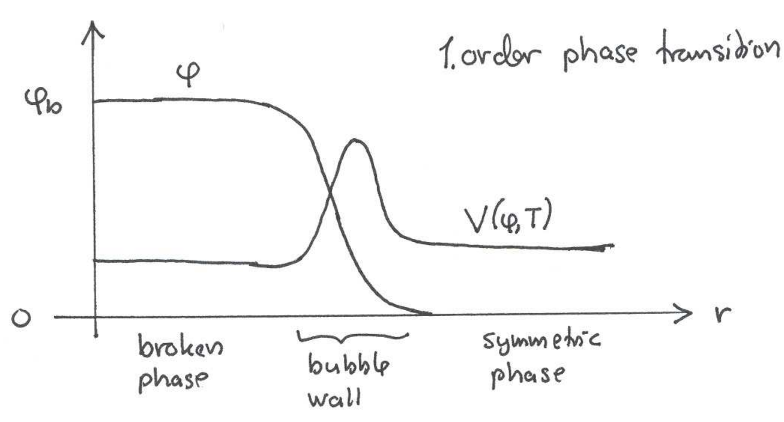


Figure 4: Field configuration for a bubble in a first-order phase transition.

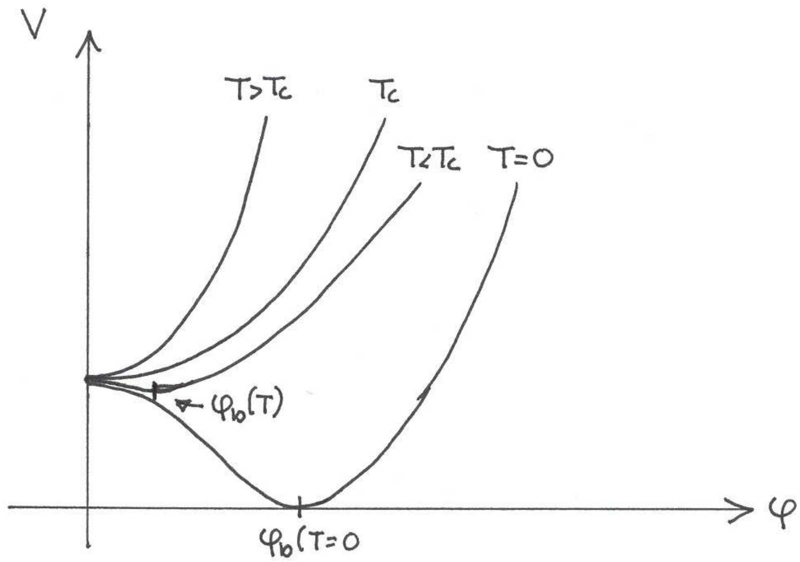


Figure 5: The effective potential for a second-order phase transition.

released in the phase transition. Here

$$\begin{aligned}\rho_{\rm s}(T) &= \rho(0,T)\\ \rho_{\rm b}(T) &= \rho(\varphi_{\rm b}(T),T)\end{aligned} \tag{27}$$

are the energy densities of the two phases. Since

$$\rho(\varphi,T) \;=\; V(\varphi,T)+Ts(\varphi,T) \;=\; V(\varphi,T)-T\frac{\partial V}{\partial T} \tag{28}$$

and at the critical temperature $V(0,T_{\rm c}) = V(\varphi_{\rm b},T_{\rm c})$, we find that

$$L = T_{\rm c}\left[\frac{\partial V}{\partial T}(\varphi_{\rm b},T_{\rm c}) - \frac{\partial V}{\partial T}(0,T_{\rm c})\right]. \tag{29}$$

In a second order phase transition there are no metastable states. When $T > T_c$, the only local minimum is $\varphi = 0$, and for $T < T_c$ it is no longer a minimum. See Fig. 5. The broken minimum $\varphi_b(T)$ exists only for $T < T_c$ and for $T \to T_c$, $\varphi_b(T) \to 0$. For a second-order phase transition the latent heat is zero, and the transition takes place instantaneously.

The effective potential of the symmetry breaking scalar field described above (Eqs. 20 and 21) is

$$V(\varphi,T) = V_0 - \frac{1}{2}\mu^2\varphi^2 + \frac{1}{4}\lambda\varphi^4 - \frac{1}{24}\mu^2T^2 + \frac{1}{8}\lambda\varphi^2T^2 - \frac{\pi^2}{90}T^4 \tag{30}$$

(ignoring quantum corrections). We find the local minima from the condition

$$\frac{\partial V}{\partial\varphi} = 0 \quad\Rightarrow\quad \begin{cases}\varphi = 0 & \text{(symmetric phase)}\\ \varphi = \varphi_{\rm b} = \sqrt{\dfrac{\mu^2}{\lambda} - \dfrac{1}{4}T^2} & \text{(broken phase)}\,.\end{cases} \tag{31}$$

The mass of the Higgs particle is given by

$$m_{\mathrm{b}}^2(T) \;=\; \frac{\partial^2 V}{\partial\varphi^2}(\varphi_{\mathrm{b}}, T) \tag{32}$$

in the broken phase and by

$$m_{\mathrm{s}}^2(T) \;=\; \frac{\partial^2 V}{\partial\varphi^2}(0, T) \;=\; -\mu^2 + \frac{1}{4}\lambda T^2 \tag{33}$$

in the symmetric phase. (If we are at a minimum, these expressions are positive.)

Exercise: Show that this leads to a second-order phase transition.